GCE: Analysis, measure theory, Lebesgue integration

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Exercise 1:

Construct a subset $A \subset \mathbb{R}$ such that A is closed, contains no intervals, is uncountable, and has Lebesgue measure $1/2$ (i.e. $|A| = 1/2$). Also explain why your set A has each of the above properties.

Hint: One possible approach here is to adjust the construction of the Cantor set to achieve a Cantor-like set with measure $1/2$, but you don't need to have seen the Cantor set to answer the question.

Exercise 2:

(i). Let $(X, \mathcal{A}, \mu)$ be a measure space, and f_n a sequence in $L^1(X)$. Let f be in $L^1(X)$. Assume that $\int f_n$ converges to $\int f$, f_n converges to f almost everywhere, and for each n, $f_n \geq 0$, almost everywhere. Show that f_n converges to f in $L^1(X)$.

Hint: Set $g_n = \min(f_n, f)$. Note that $|f_n - f| = f + f_n - 2g_n$.

(ii). Find a sequence f_n in $L^1(\mathbb{R})$ and f in $L^1(\mathbb{R})$ such that $\int f_n$ converges to $\int f$, f_n converges to f almost everywhere, but f_n does not converge to f in $L^1(\mathbb{R})$.

Exercise 3:

(i). Let f be in $L^1([0, \infty))$. Show that

$$\lim_{x \to 0^+} \int_0^\infty f(t) e^{-xt} dt = \int_0^\infty f(t) dt$$

(ii). Let $[a, b]$ be an interval in $\mathbb{R}$. If $\tilde{f}$ is continuous on $[a, b]$, g is differentiable on $[a, b]$ and monotonic, and g' is continuous on $[a, b]$, we can prove that there is a c in $[a, b]$ such that

$$\int_a^b \tilde{f} g = g(a) \int_a^c \tilde{f} + g(b) \int_c^b \tilde{f}$$

Using this result, show that if g is as specified above and f is in $L^1([a, b])$, there is a c in $[a, b]$ such that

$$\int_a^b f g = g(a) \int_a^c f + g(b) \int_c^b f$$

(iii). Let f be in $L^\infty([0,\infty))$. Assume that there is a constant L in $\mathbb{R}$ such that $\lim_{x\to\infty}\int_0^x f = L$. Show that

$$\lim_{x\to 0^+}\int_0^\infty f(t)e^{-xt}dt = L$$